J80-140

Comment on "Direct Solutions for Strum-Liouville Systems with Discontinuous Coefficients"

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HODGES[1,2] has presented a powerful technique for solving this class of problems based upon using separate modal series for each interval within which the otherwise discontinuous coefficients are smoothly varying functions. The method is basically a Ritz approach using simple polynomials in the modal series and (geometric) constraint conditions to enforce the proper physical conditions between intervals. These constraint conditions are used to eliminate certain of the modal coefficients in the series. One thus obtains a reduced problem of the usual modal type whose order is that of the total number of terms in all the modal series minus the number of constraints.

The purpose here is to note that problems of this type have been treated inter alia in the context of component mode analyses and to discuss some of the options the analyst has in considering such problems. The reader's attention is drawn to Refs. 3 and 4, the former being most directly relevant to the present discussion and the latter containing an extensive references list of broadly related literature.

In the approach of Klein,[3] modal expansions are again made in each interval. However free-free normal modes rather than polynomials are used. These normal modes (which are known analytically for uniform beams) are convenient because of their orthogonality properties, but they do lack the basic simplicity of polynomials. For structures where the free-free normal modes are not readily known (e.g., plates rather than beams), products of beam free-free modes have been proven effective in similar analyses.[5] Of course, one can use polynomials or other primitive modes to compute the free-free normal modes for a given interval.

Another difference between Refs. 1, 2, 3, and 4 is in the use of the constraint conditions. In Refs. 1 and 2 these are used to eliminate a number of modal coefficients equal to the number of constraints. By contrast, in Ref. 3 the constraints are taken into account by using Lagrange multipliers, thereby retaining all modal coefficients. The latter procedure increases (at least temporarily, but see remarks below) the total number of unknowns to the number of all original modal coefficients plus a number of Lagrange multipliers equal to the number of constraint conditions. It has the advantage of treating all modal coefficients on an equal footing and also avoids any theoretical possibility of forming an ill-conditioned matrix by an inadvertent choice of modal coefficients for elimination by the constraint conditions. The Lagrange multipliers also supply physical information, i.e., the forces of constraint. The eigenvalue problem resulting from the use of Lagrange multipliers as in Ref. 3 has been solved successfully directly.[6] However, as Klein[3] and Dowell[4] have emphasized, because of the orthogonality (in each interval) of the assumed free-free normal modes, one may eliminate analytically all of the modal coefficients by solving for them in terms of the Lagrange multipliers. This reduces the order of the eigenvalue problem to one equal to the number of constraints. The reduced eigenvalue matrix is not of the usual form (having poles at each natural frequency of each free-free component), but it is readily solved numerically. It also provides a means of obtaining bounds on the eigenvalues of the total system.[4]

Whether the analyst will find the method of Refs. 1 and/or 3 and 4 more effective for a given application is a highly subjective decision. Perhaps the point which should be emphasized (as Hodges[1,2] and Klein[3] have noted) is that Refs. 1, 2, and 3 provide a method of generating finite elements of variable order by selecting a variable number of modes in each interval. They thereby produce a highly flexible and versatile extension to the finite element method. In particular, this method generates in a very efficient manner higher order finite elements, which allow the use of a small number of elements (components) in regions with smoothly varying system properties. Such elements provide more accurate solutions than a larger number of low order elements. Indeed, for the former convergence is always assured as the order of the element increases, while for the latter convergence is often much slower and sometimes nonexistent. In general, one wishes to use the smallest number of elements possible which accurately represents the variation of system properties such as mass and stiffness. The approach of Ref. 3 is advantageous in this respect in that the cost of computation is determined primarily by the number of elements, and is virtually independent of the number of degrees of freedom per element.

Acknowledgment

The writer would like to thank Dewey H. Hodges for several valuable discussions which improved the accuracy and readability of these comments.

no ref (TN) J79-002

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Reply by Author to Earl H. Dowell

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I APPRECIATE Prof. Dowell's interest and comments on Refs. 1 and 2 and have only three points to add. First, there is additional complexity in the analysis of general structures by element (i.e., component) normal modes with Lagrange multipliers.[3] Unfortunately, this increase in complexity does

Received Oct. 30, 1979; revision received Jan. 7, 1980.

Index categories: Structural Dynamics; Heat Conduction; Vibration.

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Received Jan. 2, 1980. This paper is declared a work of the U.S. Government and therefore is in the public domain.

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not necessarily result in an increase in accuracy over the use of simpler element admissible functions for a given number of degrees-of-freedom in the complete structure. The approach of Refs. 1 and 2 uses terms of a power series as element-admissible functions which may be suitably arranged for a finite element type analysis with the added versatility that the number of displacement functions is variable within the element. Admissible polynomials are easily obtained and integration over the element can be performed exactly and simply by Gaussian quadrature. Constraints are easily expressed in terms of the polynomial coefficients. This seems to be simpler than the approach of Ref. 3, in which the element free-free normal modes are first obtained and thus used as element displacement functions. While there appears to be a theoretical possibility that an inadvertent choice of modal coefficients for elimination could lead to numerical difficulties with the approach of Refs. 1 and 2, this has not been observed in practice.

Second, the use of free-free modes for element displacement functions may actually hinder convergence because of the vanishing second and third derivatives of displacement at the element boundaries. It is interesting to compare the rate of convergence reported in Refs. 1 and 2 with that of Ref. 3. It is evident that the natural frequencies converge much faster when the simple admissible polynomials are used.[1,2] Moreover, the bending moment and shear force distributions in Ref. 2, obtained from derivatives of displacements, converge to the exact solution at *all* points along the beam as the number of terms in each element is increased. On the other hand, the method of Ref. 3 has an artificial constraint imposed because of the use of free-free modes which possess zero bending moment and shear force at the element boundaries. The resulting derivatives of displacements, while being reasonably accurate away from element boundaries, cannot converge to the exact solution at the boundaries themselves. In fact, unlike the results of Refs. 1 and 2, many modes are needed to achieve even a reasonably accurate estimate of the bending moment and shear force distribution near element boundaries without resorting to integration of applied and inertial loads. It should be noted, however, as already stated by Prof. Dowell, that the computation time in Ref. 3 is virtually independent of the number of modes per element and is primarily a function of the number of elements.

While no definitive proof is yet available that demonstrates the superiority of one method over the other, it is hoped that these observations (along with Prof. Dowell's) will help the reader better understand the relationship between the two methods.

Finally, I offer some remarks on the practical aspects of the increased versatility of both methods over conventional finite element methods. Rather than having to generate a new element geometry to check the accuracy of a solution, it is much simpler to add more degrees-of-freedom per element. Moreover, results from both methods, indicate that convergence is more rapid when the number of elements is held fixed while the number of degrees of freedom per element is increased. Thus, to model a structure in the optimum way the element boundaries are placed at points of rapid changes in stiffness or mass properties and other "discontinuities." This implies that a structure with smoothly varying properties can be modeled most efficiently as one element with a variable number of degrees-of-freedom.

References

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